Multivariate Determinants
Through Univariate Interpolation

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Abstract:

Determinants are not the easiest or quickest calculation in any numerical software package. Symbolic determinants only increase the difficultly. For the most part we have efficient algorithms for calculating determinants of numerical matrices. We have good algorithms for calculating the determinants of matrices with univariate elements. However there are few (simple) algorithms for calculating the determinants of matrices involving multivariate polynomials. In this paper we will present an easy to implement, accurate and robust method for calculating the determinant of a matrix with multivariate polynomials as entries. It is not the fastest method, but it will always work. This method will rely on the existence of an efficient algorithm to calculate numerical determinants and an efficient algorithm to interpolate polynomials of large degree. Both of which are known to exist and are available from a variety of sources.

1 Introduction

Within this paper will be found an easy, accurate and robust method for calculating the determinant of a matrix which contains multivariate polynomial entries. The particular motivation for this paper was finding a way to calculate the resultants necessary for transforming parametric surface representations into implicit representations. However that is only one potential use for the algorithm we will present. The need for such an algorithm extends across various engineering and scientific fields [Gasc2000] and is lacking in general theory [Olve2004]. Most papers have focused on divided difference and subdivision approaches and usually have a rather heavy theoretical presentation of their methods [Debo1990], [Gasc1999], [Gusk1998], [Saue1995]. Some others have been concerned with specific circumstances [Marc2001], [Marc2002]. While we would hope this paper assists in the development of a general theory, we are much more concerned with a direct and immediate application. We will also be deviating from the majority of the papers in methodology and presentation for our method is based on an algebraic trick proposed in [Moen1976], and rather than focus on the theoretical discussion of why this trick works we will be more concerned with presenting how it works. This presentation will be geared towards a single application, that of finding the determinant of a multivariate matrix.

The layout of this paper is brief. There is no theoretical discussion on this method. In section 2 we present the requirements for using the method along with a description of the method itself. In section 3 we present a couple examples demonstrating the method. Section 4 offers a short summary and conclusion.

2 The Method

2.1 The Requirements

To use the method we are about to describe several assumptions are made:

1. The symbolic matrix under consideration is nonsingular.
2. A fast method to calculate the determinant of a numerical matrix exists.
3. A fast method to perform a univariate interpolation on a large set of data points exists.
4. The number of variables in the determinant is known.
5. The maximum degree of each variable in the determinant can be found.

2.2 The First Step

Since we have assumed the matrix is nonsingular and the number of variables found in the determinant is known, the first piece of information to calculate is the maximum degree of each variable in the determinant. A general way to do this can be found in [Ding2004] and for cases involving parametric surfaces [Marc2002] may be helpful. Once these degree values are determined we can move on to the next step.

But first, some notation, assume there are n variables named x_i, for $i = 1$ to n. Denote the maximum degree of each x_i by d_i. With this notation in mind, we move to the key step.

2.3 The Trick

The beauty of this method is in a "substitution" trick. As we are not dealing with an equality we need not concern ourselves with the actual relationship between the variables of the determinant. What we do care about is finding the polynomial that represents the determinant. So in considering this method it may be best to draw an analogy found in calculus. Specifically when integrating complicated functions it is often advantageous to perform a "U-substitution" to allow us to apply "simpler" rules. The actual 'U' has little meaning, it exists solely as a placeholder with nice properties and in the end it vanishes.

2.3.1 The Kroenecker Trick

In the paper [Moen1976] a method for multiplying multivariate polynomials is presented that is referred to as the "Kroenecker Trick Algorithm." This trick effectively reduces the problem of multivariate polynomial multiplication into a problem of univariate polynomial multiplication. It is summarized as follows:

Substitution and Multiplication Step

Given two (multivariate) polynomials:

1. Let n be the number of unique variable names.
2. Find the highest degree d_i of each variable x_i across both polynomials.
3. For $i = 1$ to $n - 1$
   a. Substitute $(x_{i+1})^{d_{i+1}+1}$, that is x_i raised to the $(d_{i+1} + 1)$, in for x_i.
      Note: this results in each x_i equaling x_n raised to some power, denote this power p_i. And by definition let $p_0 = 0$.
4. There are now two univariate polynomials, multiply them using a fast univariate multiplication routine.

Recovery Step

Recover the multivariate solution by performing the following:

1. For each x_i, $i = 1$ to $n - 1$.
   a. For $k = 0$ to d_i
      i. Locate the coefficient of the x_i^k by isolating the terms of degree less than $(k+1)*p_i$. Divide these terms by $(x_n)^{k*p_i}$ and multiply by x_i^k.

While this trick may seem complicated it is quite simple, as the examples in section 3 will illustrate.

2.3.2 Reapplying the Kroenecker Trick

For our case of calculating a determinant we do not have a given polynomial. However we do have the maximum degrees of each variable that will be in the polynomial we desire. Applying the substitutions to the matrix entries as described above and assuming the monomial $x_1^{d_1} * x_2^{d_2} * \ldots * x_n^{d_n}$ exists in the determinant then the univariate form of our polynomial would have a maximum degree of

$$D = \sum_{i=1}^{n}\left(d_i * \prod_{k=i+1}^{n} (d_k + 1) \right)$$

Equation 1

While this bound is often (much) larger than needed it is the absolute worst case and thus will apply to all cases.

However there are some easy ways to reduce it. The first is to consider that we have performed a substitution that makes our matrix under consideration to be univariate. To determine the maximum degree of its determinant we may apply the univariate case of [Ding2004] or apply the method described in [Henr1999] or apply any other appropriate method. So for specific applications it would be recommended to perform some analysis on the particular matrix being studied with an effort towards reducing this bound. It should be understood that much of the research already done has in fact been geared toward discovering such limits and there are a papers for a variety of such cases [Marc2001], [Marc2002]. Regardless once this D has been determined we know we will need $D + 1$ interpolation points to evaluate it.

2.4 Finishing It

As we have now determined how many points we need, and we have performed the substitutions as described above on the matrix entries, we may now use our fast method for calculating the determinant of numerical matrices. Specifically we must select $D + 1$ values to put into the matrix for x_n. (*Notice we may need to be careful in the range of such values as the limitations of built-in numeric types will likely be a factor*.) Having performed these calculations for each entry in the matrix we then calculate the $D + 1$ numeric values of the determinant. This gives us $D + 1$ interpolation points. From here we apply our favorite univariate interpolation routine and we will arrive at a univariate polynomial that represents our determinant. Of course we “know” the determinant should be a multivariate polynomial. To recover this polynomial we simply perform the recovery step as described above and we are done.

3 Some Examples

In this section we will present a couple examples illustrating how to apply our method. The exact details of finding the degree of each variable [Ding2004], [Henr1999], [Marc2002], calculating a numeric determinant and performing an interpolation are omitted for brevity and because there are plenty of descriptive sources for such tasks [Press2002].

3.1 A Simple Example

Let $A = \begin{bmatrix} x+1 & 0 & 3y^2 \\ 1 & x & xy \\ 0 & x^2+1 & 2y \end{bmatrix}$.

Whose determinant can be found by hand to be: $-x^4y - x^3y + 3x^2y^2 + x^2y + xy + 3y^2$. So we "know" the maximum degree of x is 4 and the maximum degree of y is 2. While we could apply the method described in [Ding2004] to obtain these bounds, for brevity we will simply use the 4 and 2 and pretend we derived them. Likewise we will pretend the monomial x^4y^2 exists in the determinant (though we "know" it does not).

So we let $x = y^3$ and substitute into A to obtain $A' = \begin{bmatrix} y^3+1 & 0 & 3y^2 \\ 1 & y^3 & y^4 \\ 0 & y^6+1 & 2y \end{bmatrix}$.

We will now need to calculate the numerical determinant several times. Specifically, applying Equation 1:

$$D = \sum_{i=1}^{n}\left(d_i * \prod_{k=i+1}^{n}(d_k+1)\right) = 4 + 4*(2+1) = 4 + 12 = 16$$

we see the maximum degree of the determinant would be 16, so we need at most 17 points. We could also apply the univariate method of [Ding2004] to obtain D = 13, thus requiring only 14 points. We mention this only to illustrate the ease with which the method could be improved and will continue using the 17.

Picking 17 values, perhaps using $y = \{ -8, -7, -6, \ldots, 0, \ldots, 7, 8 \}$ we obtain 17 values for the determinant. This gives us 17 points to use in a "fast" interpolation method and we discover $\det(A') = -y^{13} - y^{10} + 3y^8 + y^7 + y^4 + 3y^2$.

We now recover the multivariate result, by "dividing out powers of y." To obtain the coefficients of x^0, we would first need to locate all the terms (monomials) with degree less than 3 $=\{ 3y^2 \}$. We then divide each of these term(s) by y^0 and multiply by x^0. Which gives us, $3y^2$.

Next, to find the coefficients of x, we locate all the terms with degree less than 6 = { y^4 }. We then divide each of these terms(s) by y^3 and multiply by x. Which gives us xy.

To obtain the coefficients of x^2, we locate all the terms with degree < 9 = { $3y^8$, y^7 } We then divide each of these terms by y^6 and multiply by x^2. Which gives us $3x^2y^2$, x^2y.

To obtain the coefficients of x^3, we now locate all the terms with degree < 12 = { $-y^{10}$ }. Dividing these term(s) by y^9 and multiplying by x^3 we obtain $-x^3y$.

Finally, to find the coefficients of x^4, we locate all the terms with degree < 15 = { $-y^{13}$ }. Dividing these term(s) by y^{12} and multiplying by x^4 we obtain $-x^4y$.

And we now have our multivariate determinant:

$$\det(A) = 3y^2 + xy + 3x^2y^2 + x^2y - x^3y - x^4y.$$

3.2 A More Complex Example

Let $A = \begin{bmatrix} x+1 & x^2z & 3y \\ 7 & x^3+y & xy \\ y^2+y+1 & x & x^2+2y \end{bmatrix}$

Notice the determinant of this matrix can be calculated by hand to be:

$$\begin{aligned} &x^6 + x^5 + 2x^4y - 7x^4z \\ &\quad + x^3y^3z - 3x^3y^3 + x^3y^2z - 3x^3y^2 + x^3yz - x^3y \\ &\quad\quad - 14x^2yz \\ &\quad\quad\quad + 2xy^2 + 21xy \\ &\quad\quad\quad\quad - 3y^4 - 3y^3 - y^2 \end{aligned}$$

So we "know" the highest degree of x is 6, the highest degree of y is 4 and the highest degree of z is 1. We will pretend we derived these degrees in some fashion [Ding2004]. We will also pretend the monomial x^6y^3z could exist in the determinant. Because of this we make the following substitutions:

$$y = z^2$$
$$x = y^5 = z^{10}$$

Thus our univariate matrix is:

$$A' = \begin{bmatrix} z^{10}+1 & z^{21} & 3z^2 \\ 7 & z^{30}+z^2 & z^{12} \\ z^4+z^2+1 & z^{10} & z^{20}+2z^2 \end{bmatrix}$$

By equation 1, D = 6*(4+1)*(1+1) + 4*(1+1) + 1 = 69 (again we could apply the univariate method of [Ding2004] to obtain D = 60, but will again continue using the 69). Thus the maximum degree of the determinant is 69. Because of this we will need to find 70 values of z at which to evaluate the det(A'). Notice our selection of values may be limited by the range of the data types of whatever programming language with which we might choose to implement this. Overcoming that limitation may be difficult in some situations. However, assuming we successfully obtain 70 determinant values we will have 70 points on which to perform an interpolation. This interpolation would yield the following result:

$$
\begin{aligned}
&z^{60} + z^{50} \\
&\quad + 2z^{42} - 7z^{41} \\
&\quad\quad + z^{37} - 3z^{36} + z^{35} - 3z^{34} + z^{33} - z^{32} \\
&\quad\quad\quad - 14z^{23} \\
&\quad\quad\quad\quad + 2z^{14} + 21z^{12} \\
&\quad\quad\quad\quad\quad - 3z^{8} - 3z^{6} - z^{4}
\end{aligned}
$$

Taking all the terms where the degree is < 10 = { $-3z^8$,$-3z^6$, $-z^4$ }, then dividing each by z^0 and multiplying by x^0, we get $-3z^8 - 3z^6 - z^4$.

Taking all the terms where the degree is < 20 = { $2z^{14}$, $21z^{12}$ }, then dividing each by z^{10} and multiplying by x, we get $(2z^4 + 21z^2) * x$.

Taking all the terms with degree < 30 = { $-14z^{23}$ }, then dividing each by z^{20} and multiplying by x^2, we arrive at $(-14z^3) * x^2$.

Taking all the terms with degree < 40 = { z^{37}, $-3z^{36}$, z^{35}, $-3z^{34}$, z^{33}, $-z^{32}$ }, then dividing each by z^{30} and multiplying by x^3, we arrive at $(z^7 - 3z^6 + z^5 - 3z^4 + z^3 - z^2) * x^3$.

Continuing we take all the terms where the degree is < 50 = { $2z^{42}$, $-7z^{41}$ }, then divide by z^{40} and multiply by x^4, we would end with $(2z^2 - 7z)*x^4$.

In a like manner we also obtain $(1)*x^5$ and $(1)*x^6$.

We now need to recover the y coefficients. So for each of the coefficients of the various powers of x (i.e. x^0, x^1, x^2, x^3, x^4) we would need to perform the following:

For the coefficient of x^0 which was determined above to be $-3z^8 - 3z^6 - z^4$.

> We first find the coefficient of y^0 by locating all the terms where z has a degree less than 2. In this case there are none.
>
> Next we find the coefficient of y by locating the terms with degree less than 4 which again, in this case, there are none.
>
> Then we find the coefficient of y^2 by isolating the terms with degree less than 6 which gives us { $-z^4$ }. We then divide by z^4 and multiply by y^2 to get $-y^2$.

We next find the coefficient of y^3 by locating the terms with degree less than 8 which yields { $-3z^6$ }. We then divide by z^6 and multiply by y^3 to get $-3y^3$.

Finally we find the coefficient of y^4 by locating the terms with degree less than 10 which yields { $-3z^8$ }. We then divide by z^8 and multiply by y^4 to get $-3y^4$.

For the coefficient of x which was derived above to be $2z^4 + 21z^2$.

We find the coefficient of y^0 by locating all the terms where z has a degree less than 2. In this case there are none.

Next we find the coefficient of y by locating the terms with degree less than 4 which gives us { $21z^2$ }. We then divide by z^2 and multiply by y to get $21xy$.

Then we find the coefficient of y^2 by isolating the terms with degree less than 6 which gives us { $2z^4$ }. We then divide by z^4 and multiply by y^2 to get $2xy^2$.

We next find the coefficient of y^3 by locating the terms with degree less than 8 Which in this case, there are none.

And finally we find the coefficient of y^4 by locating the terms with degree less than 10 which, again in this case, are none.

And we would do likewise for the coefficients of x^2, x^3, x^4, x^5 and x^6 to eventually arrive at the determinant of A to be:

$$\det(A) = x^6 + x^5 + 2x^4y - 7x^4z + x^3y^3z - 3x^3y^3 + x^3y^2z - 3x^3y^2 + x^3yz - x^3y - 14x^2yz + 2xy^2 + 21xy - 3y^4 - 3y^3 - y^2$$

4 Summary

So we have now presented an easy to implement, robust method for calculating the determinant of a multivariate matrix using already existing univariate methods. We have also shown several examples demonstrating how this method works. While this method may not always be the most efficient, it will always work.

To improve the speed of this method we would strongly recommend using the univariate method described in [Ding2004] or the method described in [Henr1999] to evaluate the maximum bound D. With that said we readily acknowledge that the more information that can be predetermined about the determinant, such as which monomials exist in the determinant and the degree of each variable in each such monomial, would serve to possibly reduce the number of interpolation points necessary. This would, in turn, improve the speed of this method. However, it is not necessary for the method to work. The upper bound, D, as described is the worst case bound.

While there are other improvements that could be made, the above method is an easy to implement, robust method for calculating the determinant of multivariate matrices. It has the advantages of requiring very little knowledge of advanced algebra and requiring only an understanding of the (already well developed) univariate methods. It also eliminates the need for a complex multivariate computer algebra system, which is useful if the desire is to solve a particular problem in a short amount of time. Further this method has been implemented and does perform faster, in significantly large cases, than calculating the determinant through direct polynomial multiplications, additions and subtractions.

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