Fall 1998 Math 151 Common Exam 1

c 1998, Art Belmonte, Math 151, Fri, 02/Oct/98

Multiple Choice Answers

A1-b, B10-b

```
> Limit(1/(x-1) - 2/(x^2-1), x=1); answer:=value(%);
```

$$\lim_{x \to 1} \frac{1}{x-1} - 2\frac{1}{x^2-1}$$

$$answer := \frac{1}{2}$$

A2-c, B5-c

```
> Limit((sqrt(x^2+8)-3) / (x-1), x=1); answer:=value(%);
```

$$\lim_{x \to 1} \frac{\sqrt{x^2+8}-3}{x-1}$$

$$answer := \frac{1}{3}$$

A3-b, B3-b

```
> y = Limit((x-sqrt(x^2+4*x-1)) / 2, x=infinity); horizontal_asymptote:=value(%);
```

$$y = \lim_{x \to \infty} \frac{1}{2}x - \frac{1}{2}\sqrt{x^2+4x-1}$$

$$horizontal\_asymptote := y = -1$$

As we see, there is no horizontal asymptote in the other direction.

```
> Limit((x-sqrt(x^2+4*x-1)) / 2, x=-infinity); only_one:=value(%);
```

$$\lim_{x \to (-\infty)} \frac{1}{2}x - \frac{1}{2}\sqrt{x^2+4x-1}$$

$$only\_one := -\infty$$

A4-d, B8-d

From *Figure A* (drawn on the exam), we have that $\lim_{x \to 1+} f(x) = 1 = f(1)$. Therefore, *f* is continuous from the right at $x = 1$.

A5-e, B7-e

From *Figure A* (drawn on the exam), we see that none of the first four statements (a)-(d) is true. Hence the correct choice is (e). Here are details.

- Statement (a) is *false* since *f* has a (removable) discontinuity at $x = 0$. Since differentiability at at point implies continuity thereat, f *cannot* be differentiable at $x = 0$.
- Statement (b) is *false* since *f* has a (jump) discontinuity at $x = 1$. Since differentiability at at point implies continuity thereat, *f* *cannot* be differentiable at $x = 1$.
- Statement (c) is *false* since the graph of *f* is "sharp/pointed/kinked" at $x = 2$. The geometric connotation of differentiability is that of "smoothness." The slope of the tangent line (from *Figure A*) cannot simultaneously be both positive and negative.
- Statement (d) is *false* since *f* has an (infinite) discontinuity at $x = 4$. Since differentiability at at point implies continuity thereat, f *cannot* be differentiable at $x = 4$.

A6-a, B2-a

```
> unassign('s', 't'); a:=[2,-1]; b:=[3,2]; c:=[5,3];
  eq:=equate(c, s*a + t*b); solve(eq, {s,t}); assign(%);
  t_over_s:=t/s;
```

$a := [2, -1]$

$b := [3, 2]$

$c := [5, 3]$

$eq := \{5 = 2\,s + 3\,t, 3 = -s + 2\,t\}$

$\{t = \frac{11}{7}, s = \frac{1}{7}\}$

$t\_over\_s := 11$

A7-d, B11-d

The vector sum of **b** and **c** is **a**, as the head-to-tail diagram indicates. Now subtract **b** from each side of this vector equation.

```
> unassign('a', 'b', 'c');
  b + c = a; map(sort, op(solve(%, {c})));
```

$b + c = a$

$c = a - b$

A8-e, B4-e

```
> x^5 / (x^3 - 2); derivative:=simplify(diff(%, x));
  equivalently:=expand(numer(%)) / denom(%);
```

$\frac{x^5}{x^3 - 2}$

$derivative := 2\,\frac{x^4\,(x^3 - 5)}{(x^3 - 2)^2}$

$equivalently := \frac{2\,x^7 - 10\,x^4}{(x^3 - 2)^2}$

A9-e, B1-e

The slope m of the tangent line at $x = 1$ is the derivative $f\,'(1)$. Then use the point-slope formula.

```
> f:=x->x^3 - x; m:=D(f)(1);
  point_slope_formula:=y - f(1) = m*(x - 1);
```

$f := x \rightarrow x^3 - x$

$m := 2$

$point\_slope\_formula := y = 2\,x - 2$

A10-b, B6-b

Here is the distance from the point to the line computed via vectors. We make use of the **`vec_calc`** routines **`dot`** and **`len`**, which respectively compute the length of a vector and the dot product between two vectors. (We have autoloaded the **`vec_calc`** package via a Unix *.mapleinit* file in our home directories.)

```
> unassign('t'); P:=[-2,3]; L:=[t->4*t, t->3*t + 5/4];
  a:=D(L); b:=P - L(1/4);
  distance:=len(b  -  dot(a,b)/len(a) * a/len(a));
```

$P := [-2, 3]$

$L := \left[t \rightarrow 4\,t, t \rightarrow 3\,t + \frac{5}{4}\right]$

$a := [4, 3]$

$b := [-3, 1]$

$distance := \frac{13}{5}$

A11-c, B9-c

Parallel translate the vectors so that they emanate from the same starting point; i.e., put their tails together. The vector between their heads is their difference vector. Accordingly, since these three vectors are *unit* vectors, an *equilateral* triangle is formed with each side 1 unit long. With this crucial piece of geometry dispatched, position the starting point at the origin and rotate to taste. Then compute the needful: *voila!*

```
> a:=[1,0]; b:=[1/2, sqrt(3)/2]; length_of_vector_sum:=len(a+b);
```

$a := [1, 0]$

$b := \left[\frac{1}{2}, \frac{1}{2}\sqrt{3}\right]$

$length\_of\_vector\_sum := \sqrt{3}$

Work-Out Solutions

A12/B17

The *continuous* polynomial function $f(x) = x^4 + x^3 + x - 1$ has *values* $f(0) = -1 < 0$ and $f(1) = 2 > 0$, which are of *opposite sign*. Hence $f(x) = 0$ for some x in the *interval* [0, 1] by the *Intermediate Value Theorem*.

A13/B15

With the positive x-axis pointing east and the positive y-axis pointing north, we have $\mathbf{v}_P = < 900 \cos(60^o), 900 \sin(60^o) > = < 450, 450\sqrt{3} >$ and $\mathbf{v}_W = < 100, 0 >$. Thus $\mathbf{v}_G = \mathbf{v}_P + \mathbf{v}_W = < 550, 450\sqrt{3} >$. Therefore, $\|\mathbf{v}_G\| = \sqrt{550^2 + (450\sqrt{3})^2} = 953.94$ (km/hr) and $\theta = \arctan\left(\frac{450\sqrt{3}}{550}\right) = 54.8^o = 0.96$ radians, counterclockwise from east. That is, $\mathbf{v}_G = 953.94$ km/hr at a bearing of 35.2^o. [Non-decimal (symbolic) answers are fine, although most folks will probably have decimal answers, which are more appropriate for engineering.]

A14/B18

$f'(1) = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{3x^2 - x + 7 - 9}{x - 1} = \lim_{x \to 1} \frac{3x^2 - x - 2}{x - 1} = \lim_{x \to 1} \frac{(3x + 2)(x - 1)}{x - 1} = \lim_{x \to 1} (3x+2) = 5.$

A15/B12

The *velocity* $\mathbf{v}$ is the *derivative* of the position function $\mathbf{r}(t) = < 3000\,t, 100\,t\,(20 - t) >$. Hence $\mathbf{v} = \mathbf{r}'(t) = < 3000, 2000 - 200\,t >$ and thus the velocity at time $t = 8$ is $\mathbf{v}(8) = < 3000, 400 >$, a *vector*.
The *speed* is the magnitude (length) of the velocity $\mathbf{v}$. Accordingly, at time $t = 15$ the speed is $\|\mathbf{v}(15)\| = \sqrt{3000^2 + (-1000)^2} = \sqrt{10 \times 10^6} = 1000\sqrt{10} = 3162.28$, a *scalar*.

A16/B13

With $\mathbf{a} = < x, 2 >$ and $\mathbf{b} = < 2x, -4 >$, we have that $\mathbf{a} * \mathbf{b} < 0$ implies $2x^2 - 8 < 0$, whence $x^2 < 4$ or $-2 < x < 2$.

A17/B16

With $f(x) = \begin{cases} x^2 - c & x < 2 \\ 3c & x = 2, \\ x + c & 2 < x \end{cases}$ we have $\lim_{x \to 2-} f(x) = 4 - c$, whereas $\lim_{x \to 2+} f(x) = 2 + c$. Matching these one-sided limits yields $4 - c = 2 + c$, whence $c = 1$. Thus we conjecture that $f(x) = \begin{cases} x^2 - 1 & x < 2 \\ 3 & x = 2 \\ x + 1 & 2 < x \end{cases}$ is continuous at $x = 2$. Indeed:

CONDITIONS:

1. The function f is *defined* at $x = 2$: $f(2) = 3$.
2. The *limit* $\lim_{x \to 2} f(x) = 3$ *exists*.
3. The *limiting value agrees with the function value*: $\lim_{x \to 2} f(x) = f(2)$.

CONCLUSION:
Therefore, f is *continuous* at $x = 2$.

A18/B14

(a)

The spud hits the ground *when* $h(t) = 0$ for a *positive* value of t. (It can't hit the ground before the spud gun is fired!)

```
> h:=t->200 + 25*t - 4.9*t^2;
  sol:=solve(h(t)=0, t); time_of_impact:=sol[2];
```

$h := t \rightarrow 200 + 25\,t - 4.9\,t^2$

sol := -4.328226037, 9.430266853

time_of_impact := 9.430266853

The spud hits the ground after 9.43 seconds.

(b)

Its *speed* at impact is the magnitude (absolute value) of the signed velocity **v** thereat.

```
> v:=D(h); abs(v(time_of_impact));
```

$v := t \rightarrow 25 - 9.8\,t$

67.41661516

The speed of the potato at impact is 67.4 meters/second.

(c)

The *total distance* traveled by the spud is the magnitude of the distance traveled going up plus the magitude of the distance traveled going down. At its maximum height, the velocity of the spud is zero.

```
> time_at_max_height:=solve(v(t)=0, t);
  total_distance:=2*(h(time_at_max_height) - 200) + 200;
```

time_at_max_height := 2.551020408

total_distance := 263.7755102

The total distance the spud traveled is 263.8 meters.