MATH 151
EXAM #3A SOLUTIONS
FALL 2001

PART 1

1. If $\theta = \cos^{-1}(\sqrt{3}/2)$, then $\cos\theta = \sqrt{3}/2$ and $\theta = \pi/6$. Thus,

$$\tan(\cos^{-1}(\sqrt{3}/2)) = \tan\theta = \tan(\pi/6) = 1/\sqrt{3}\ .$$

The correct answer is (d).

2. Using L'Hopital's rule

$$\lim_{x\to 0} \frac{\ln(1+x) - x}{x^2} = \lim_{x\to 0} \frac{\dfrac{1}{1+x} - 1}{2x} = \lim_{x\to 0} -\frac{1}{2(1+x)} = -\frac{1}{2}\ .$$

The correct answer is (d).

3. $$\frac{d}{dx}\sin^{-1}(2/x) = \frac{1}{\sqrt{1-(2/x)^2}}(2/x)' = \frac{1}{\sqrt{1-\dfrac{4}{x^2}}}\left(\frac{-2}{x^2}\right) = \frac{1}{\sqrt{1-\dfrac{4}{x^2}}}\left(\frac{-2}{\sqrt{x^4}}\right) = -\frac{2}{\sqrt{x^4-4x^2}}\ .$$

The correct answer is (a).

4. If $f''(x) = 4x+2$, $f'(0) = 1$ and $f(0) = -3$, then

$$\begin{aligned} f'(x) &= 2x^2 + 2x + 1 \\ f(x) &= (2/3)x^3 + x^2 + x - 3 \\ f(3) &= (2/3)(3)^3 + (3)^2 + 3 - 3 = 27 \end{aligned}$$

The correct answer is (e).

5. The Riemann sum R using four equal subintervals with right endpoints is

$$\begin{aligned} R &= f(0.5)(0.5-0) + f(1)(1.0-0.5) + f(1.5)(1.5-1.0) + f(2)(2.0-1.5) \\ &= (0.5)(0.5) + (0.2)(0.5) + (0.1)(0.5) + (0.0)(0.5) \\ &= 0.4 \end{aligned}$$

The correct answer is (a).

6. If $y = (1+3x)^{\cot(x)}$, then

$$\lim_{x\to 0} \ln y = \lim_{x\to 0} \cot(x)\ln(1+3x) = \lim_{x\to 0} \frac{\ln(1+3x)}{\tan(x)} = \lim_{x\to 0} \frac{\frac{3}{1+3x}}{\sec^2(x)} = \lim_{x\to 0} \frac{3\cos^2(x)}{1+3x} = 3$$

and

$$\lim_{x\to 0} (1+3x)^{\cot(x)} = \lim_{x\to 0} y = \lim_{x\to 0} e^{\ln y} = e^{\left(\lim_{x\to 0} \ln y\right)} = e^3 .$$

The correct answer is (c).

7. If $f(x) = x^3 - 3x^2$, then $f'(x) = 3x^2 - 6x$ and $f''(x) = 6x - 6$. Now $f'(2) = 0$ and $f''(2) = 6 > 0$. Thus, f has a relative minimum at $x = 2$.

The correct answer is (d).

8. Define $f(x) = x + 2y = x + \dfrac{100}{x}$ for $x > 0$. Then $f'(x) = 1 - \dfrac{100}{x^2}$ and $x = 10$ is the only positive critical number for $f(x)$. Now $f'(x) < 0$ if $0 < x < 10$ and $f'(x) > 0$ if $x > 10$. The minimum value of $x + 2y$ given that $xy = 50$ is $f(10) = 10 + \dfrac{100}{10} = 20$.

The correct answer is (c).

9. If $\int_0^1 f(x)\,dx = 7$ and $\int_0^1 g(x)\,dx = 2$, then

$$\int_0^1 [f(x) - 4g(x)]\,dx = \int_0^1 f(x)\,dx - 4\int_0^1 g(x)\,dx = 7 - 4(2) = -1 \ .$$

The correct answer is (c).

10.

$$\begin{aligned} F(x) &= \int_0^{\sin x} \frac{1}{1+t^4}\,dt \\ F'(x) &= \left(\frac{1}{1+(\sin x)^4}\right)(\sin x)' \\ &= \left(\frac{1}{1+(\sin x)^4}\right)\cos x \\ F'(\pi/3) &= \left(\frac{1}{1+(\sin(\pi/3))^4}\right)\cos(\pi/3) \\ &= \left(\frac{1}{1+(\sqrt{3}/2)^4}\right)(1/2) = \frac{8}{25} \end{aligned}$$

The correct answer is (d).

11.

$$\begin{aligned} y &= x^{\cos x} \\ \ln y &= (\cos x)\ln x \\ \left(\frac{1}{y}\right) y' &= (\cos x)(1/x) + (\ln x)(-\sin x) \\ y' &= y\left[-(\ln x)(\sin x) + \frac{\cos x}{x}\right] \\ &= x^{\cos x}\left[-(\ln x)(\sin x) + \frac{\cos x}{x}\right] \end{aligned}$$

The correct answer is (b).

PART 2

12. If $y(t) =$ number of bacteria at time t, then

$$y'(t) = k\,y(t)$$
$$y(0) = 10^8 .$$

The solution is $y(t) = y(0)e^{kt} = 10^8 e^{kt}$.

(a) When $t = 10$,

$$\begin{aligned} 10^8 e^{10k} &= y(10) = 1.03 \times 10^8 \\ e^{10k} &= 1.03 \\ 10k &= \ln 1.03 \\ k &= \frac{1}{10} \ln 1.03 \end{aligned}$$

(b) The population has doubled its size when

$$\begin{aligned} y(t) &= 10^8 e^{kt} = 2 \times 10^8 \\ e^{kt} &= 2 \\ kt &= \ln 2 \\ t &= \frac{\ln 2}{k} = 10 \left(\frac{\ln 2}{\ln 1.03} \right) \text{ minutes} \end{aligned}$$

13. If $g(x) = x^3 - 3x^2 - 9x$, then $g'(x) = 3x^2 - 6x - 9$. The critical numbers of $g(x)$ solve

$$\begin{aligned} g'(x) &= 0 \\ 3x^2 - 6x - 9 &= 0 \\ 3(x-3)(x+1) &= 0 . \end{aligned}$$

The solutions are $x = -1$ and $x = 3$. Now

$$\begin{aligned} g(-2) &= (-2)^3 - 3(-2)^2 - 9(-2) = -2 \\ g(-1) &= (-1)^3 - 3(-1)^2 - 9(-1) = 5 \\ g(3) &= (3)^3 - 3(3)^2 - 9(3) = -27 \\ g(4) &= (4)^3 - 3(4)^2 - 9(4) = -20 \end{aligned}$$

The function $g(x) = x^3 - 3x^2 - 9x$ has an absolute maximum value of 5 on $[-2, 4]$.

14. If $f(x) = e^{-2x}\tan^{-1}(\ln x)$, then

$$\begin{aligned} f'(x) &= e^{-2x}\left(\tan^{-1}(\ln x)\right)' + \left(e^{-2x}\right)'\tan^{-1}(\ln x) \\ &= e^{-2x}\left(\frac{1}{1+(\ln x)^2}\right)(\ln x)' - 2e^{-2x}\tan^{-1}(\ln x) \\ &= \left[e^{-2x}\left(\frac{1}{1+(\ln x)^2}\right)\frac{1}{x}\right] - 2e^{-2x}\tan^{-1}(\ln x) \end{aligned}$$

15. If $g(x) = \frac{1}{4}x^4 - \frac{1}{6}x^6$, then

(a) $g'(x) = x^3 - x^5 = x^3(1-x^2)$ and $g'(x) > 0$ on $(-\infty,-1)\cup(0,1)$ and $g'(x) < 0$ on $(-1,0)\cup(1,\infty)$. The graph of $g(x)$ is increasing on $(-\infty,-1)\cup(0,1)$ and decreasing on $(-1,0)\cup(1,\infty)$.

(b) $g''(x) = 3x^2 - 5x^4 = x^2(3-5x^2)$ and $g''(x) > 0$ on $(-\sqrt{3/5},\sqrt{3/5})$ and $g''(x) < 0$ on $(-\infty,-\sqrt{3/5})\cup(\sqrt{3/5},\infty)$. The graph of $g(x)$ is concave up on $(-\sqrt{3/5},\sqrt{3/5})$ and concave down on $(-\infty,-\sqrt{3/5})\cup(\sqrt{3/5},\infty)$.

16. (a) $F(x) = e^x$ is an antiderivative of $f(x) = e^x$. Then

$$\int_{\ln 2}^{\ln 9} e^x\,dx = F(x)\Big|_{\ln 2}^{\ln 9} = F(\ln 9) - F(\ln 2) = e^{\ln 9} - e^{\ln 2} = 9 - 2 = 7$$

(b) $F(x) = x + \ln x$ is an antiderivative of $f(x) = \frac{x+1}{x} = 1 + \frac{1}{x}$. Then

$$\int_2^5 \frac{x+1}{x}\,dx = F(x)\Big|_2^5 = F(5) - F(2) = (5+\ln 5) - (2+\ln 2) = 3 + \ln(5/2)$$

17. If a square with side x ft is cut out from each of the four corners, the volume of the box formed by bending up the sides is

$$V(x) = x(6-2x)^2 ,$$

for $0 < x < 3$. Now

$$V'(x) = -4x(6-2x) + (6-2x)^2 = (6-2x)(6-6x) = 12(3-x)(1-x)$$

and the critical values for $V(x)$ are $x = 1$ and $x = 3$. Only $x = 1$ is in the region $0 < x < 3$.

Now $V'(x) > 0$ if $0 < x < 1$ and $V'(x) < 0$ if $1 < x < 3$. Consequently, the maximum volume occurs when $x = 1$ and the resulting volume is $V(1) = 1(6-2(1))^2 = 16$.

18. Let $s(t) =$ the position of the car at time t and $v(t) =$ the velocity of the car at time t. Then

$$\begin{aligned} v'(t) &= -36 \\ v(t) &= -36t + v(0) \end{aligned}$$

and the car comes to rest when

$$\begin{aligned} v(t) &= 0 \\ -36t + v(0) &= 0 \\ t &= \frac{v(0)}{36} \text{ seconds} \end{aligned}$$

Now

$$\begin{aligned} s'(t) &= v(t) = -36t + v(0) \\ s(t) &= -18t^2 + v(0)t + s(0) \end{aligned}$$

and the distance traveled by the car during the skid was $s(v(0)/36) - s(0) = 112.5\,\text{ft}$. Thus,

$$\begin{aligned} s(v(0)/36) - s(0) &= -18\left(\frac{v(0)}{36}\right)^2 + v(0)\left(\frac{v(0)}{36}\right) \\ 112.5 &= \frac{(v(0))^2}{72} \\ v(0) &= 90 \text{ ft/sec} \end{aligned}$$