Section 3.2 (5.7)
Solutions and Hints

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for the book:
Calculus, Early Vectors
by James Stewart.

Depending on how much cooperation is going on between the Math Department and the Physics Department, you may or may not be simultaneously taught differentiation and integration techniques. However, it is not as bad as it sounds. Basically you memorize formulas like: "the derivative of sin(x) is cos(x)" then for integration you just reverse stuff: "the integral of cos(x) is sin(x)."
Anyway, you may need to solve problems from section 3.2 as well as 5.7.

12. Differentiate the given function:

$$f(u) = \frac{1-u^2}{1+u^2}$$

Before beginning notice that f(u) can be written as two functions: $f(u) = \frac{h(u)}{g(u)}$
Where $h(u) = 1 - u^2$ and $g(u) = 1 + u^2$.

Now we will apply the Quotient Rule which basically says:
"The derivative of f(u) equals:
(the derivative of the top) time (the bottom)
minus (the derivative of the bottom) times (the top)
all divided by the (bottom squared)."

So you get **Equation 1**: $f'(u) = [\ h'(u) * g(u) - g'(u)*h(u)\ ] / (g(u))^2$.

Which means you must find h'(u) and g'(u).
$h(u) = 1 - u^2 \rightarrow h'(u) = -2u$
$g(u) = 1 + u^2 \rightarrow g'(u) = 2u$

So put stuff into Equation 1:

$$\begin{aligned} f'(u) &= [\ -2u * (1 + u^2) - 2u*(1 - u^2)\ ] / (1 + u^2)^2 \\ &= [\ -2u - 2u^3 - 2u + 2u^3\ ] / (1 + u^2)^2 \\ &= -4u / (1 + u^2)^2 \end{aligned}$$

$$\boxed{\text{So } f'(u) = \frac{-4u}{(1+u^2)^2}}$$

15. Differentiate the given function:

$$y = \frac{x^2 + 4x + 3}{\sqrt{x}}$$

Let $y = f(x) = \frac{h(x)}{g(x)} = \frac{x^2 + 4x + 3}{\sqrt{x}}$.

So $h(x) = x^2 + 4x + 3$ → $h'(x) = 2x + 4$

and $g(x) = \sqrt{x} = x^{1/2}$ → $g'(x) = (1/2)*x^{-1/2} = \frac{1}{2} * \frac{1}{\sqrt{x}} = \frac{1}{2\sqrt{x}}$

And by the Quotient Rule we then know:

$$f'(x) = \frac{h'(x) * g(x) - g'(x) * h(x)}{\left(g(x)\right)^2}$$

$$= \frac{(2x+4) * \sqrt{x} - \frac{1}{2\sqrt{x}} * (x^2 + 4x + 3)}{\left(\sqrt{x}\right)^2}$$

$$= \frac{2x^{3/2} + 4x^{1/2} - \frac{x^{3/2}}{2} - 2x^{1/2} - \frac{3}{2x^{1/2}}}{x}$$

$$= \frac{2x^{1/2} + \frac{3x^{3/2}}{2} - \frac{3}{2x^{1/2}}}{x}$$

$$= \left(2x^{1/2} + \frac{3x^{3/2}}{2} - \frac{3}{2x^{1/2}}\right) * \frac{1}{x}$$

$$= \frac{2\sqrt{x}}{x} + \frac{3x\sqrt{x}}{2x} - \frac{3}{2x\sqrt{x}}$$

$$\boxed{f'(x) = \frac{2}{\sqrt{x}} + \frac{3\sqrt{x}}{2} - \frac{3}{2x\sqrt{x}}}$$

30. Differentiate the given function:

$$v = \frac{6}{\sqrt[3]{t^5}}$$

For this one I will first apply the Quotient Rule and then I will demonstrate a faster method based on simply rewriting the equation using the laws of exponents.

Let f(x) = v $= \frac{6}{\sqrt[3]{t^5}}$, rewrite this slightly

$$= \frac{6}{t^{5/3}}$$

Let h(x) = 6 → h'(x) = 0

and g(x) = $t^{5/3}$ → g'(x) = (5/3)*$t^{5/3-1}$ = (5/3)*$t^{2/3}$.

So f(x) = h(x) / g(x)

Thus f '(x) = [h'(x)*g(x) – g'(x)*h(x)] / $(g(x))^2$

$$f'(x) = \frac{0 * t^{5/3} - (5/3) * t^{2/3} * 6}{\left(t^{5/3}\right)^2}$$

$$= \frac{-(30/3) * t^{2/3}}{t^{10/3}}$$

$$\boxed{= \frac{-10}{t^{8/3}} \qquad = \frac{-10}{\sqrt[3]{t^8}} \qquad = \frac{-10}{t^2 * \sqrt[3]{t^2}}}$$

Now the faster method:

Let f(x) = v $= \frac{6}{\sqrt[3]{t^5}}$, rewrite this slightly

$= \frac{6}{t^{5/3}}$, and keep going so everything is in the numerator

$= 6 * t^{-5/3}$

And in this form we can just apply the "power rule."

f '(x) = 6*(-5/3)*$t^{-5/3-1}$

= -10*$t^{-8/3}$

$$= \frac{-10}{t^{8/3}}$$

Notice this way is much faster and easier AND arrives at the same answer. Later you will learn the Chain Rule, which with this trick makes things much easier when dealing with "fractional equations."

37. Find the equation of the tangent line to the given curve at the given point.

$y = x + 4/x, \quad (2, 4)$

Rewrite y slightly so we can apply the "power rule:"

$y = f(x) = x + 4*x^{-1}$

$y' = f'(x) = 1 - 4*x^{-2}$.

Now put 2 in for x, because we need the slope of the tangent line at (2, 4).

$f'(2) = 1 - 4*2^{-2} = 1 - 4 / 4 = 1 - 1 = 0$

So the slope of the curve at the point (2, 4) is 0.

Notice that the tangent line must go through the point (2, 4) with slope = 0.

Now use the point-slope equation of a line:

$(y - y1) = m*(x - x1)$
$(y - 4) = 0*(x - 2)$
$y - 4 = 0$
$y = 4$

The tangent line of $y = x + 4/x$ at the point (2, 4) is **$y = 4$.**

38. Find the equation of the tangent line to the given curve at the given point.

$y = x^{5/2}, \quad (4, 32)$
$y' = f'(x) = (5/2)*x^{3/2}$.

Now put 4 in for x, because we need the slope of the tangent line at (4, 32).

$f'(4) = (5/2)*4^{3/2} = (5/2) * 8 = 20$

So the slope of the curve at the point (4, 32) is 20.

Notice that the tangent line must go through the point (4, 32) with slope = 20.

Now use the point-slope equation of a line:

$(y - y1) = m*(x - x1)$
$(y - 32) = 20*(x - 4)$ ← some profs accept this form as a final answer.
$y - 32 = 20x - 80$
$y = 20x - 48$

The tangent line of $y = x^{5/2}$ at the point (4, 32) is **$y = 20x - 48$**

46. For what values of x does the graph: $f(x) = 2x^3 - 3x^2 - 6x + 87$ have a horizontal tangent?

Horizontal tangent means the slope of the tangent line is zero.
Recall the equation of the slope of the tangent line is f '(x).

So we find f '(x), set it equal to zero and solve for x.

$f(x) = 2x^3 - 3x^2 - 6x + 87$
$f'(x) = 6x^2 - 6x - 6$
$= 6 * (x^2 - x - 1)$, set this equal to zero

$0 = 6 * (x^2 - x - 1)$
$0 = x^2 - x - 1$ solve using the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1 \pm \sqrt{(-1)^2 - 4*1*(-1)}}{2*1}$$
$$= \frac{1 \pm \sqrt{5}}{2}$$

$f(x) = 2x^3 - 3x^2 - 6x + 87$ has horizontal tangents at $x = \frac{1 \pm \sqrt{5}}{2}$.

47. Find the points on the curve: $f(x) = x^3 - x^2 - x + 1$ where the tangent is horizontal.

Horizontal tangent means the slope of the tangent line is zero.
Recall the equation of the slope of the tangent line is f '(x).

So we find f '(x), set it equal to zero and solve for x.

$f(x) = x^3 - x^2 - x + 1$
$f'(x) = 3x^2 - 2x - 1$
$= (3x + 1)*(x - 1)$, set this equal to zero

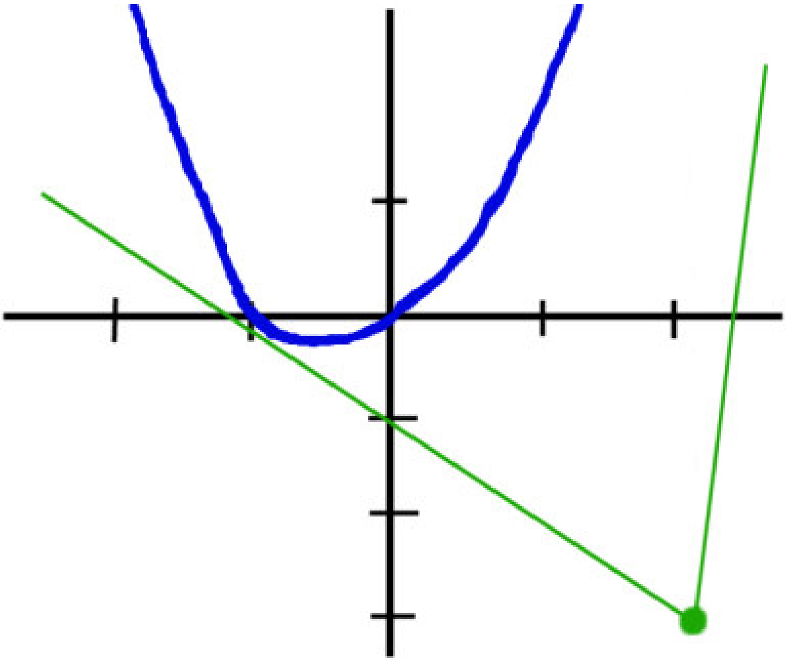
$0 = 3x + 1$ OR $0 = x - 1$
$x = -1/3$ OR $x = 1$

Now put those values of x into f(x) to get the y values of the points:
$f(-1/3) = (-1/3)^3 - (-1/3)^2 - (-1/3) + 1 = 32 / 27$
$f(1) = 1^3 - 1^2 - 1 + 1 = 0$

$f(x) = x^3 - x^2 - x + 1$ has horizontal tangents at (-1/3, 32/27) and (1, 0).

50. Find the equations of both lines through the point (2, - 3) that are tangent to the parabola $y = x^2 + x$.

Notice that (2, -3) is NOT a point on the parabola
thus there can be two tangent lines through it.
Here is a rough sketch:



The tangent lines must have a slope defined by: $y' = 2x + 1$.
And they must also go through the point (2, -3).
So they must satisfy the equation:

$$(y+3) = (2x + 1)*(x - 2)$$
$$y + 3 = 2x^2 - 3x - 2$$
$$y = 2x^2 - 3x - 5$$

Notice that the x is the same x variable as in the parabola equation.

So now we need to find the two points our tangent lines go through that are ON the parabola. To do this we set the parabola equation equal to the 'tangent line' equation:

$x^2 + x = 2x^2 - 3x - 5$, and we solve for x
$0 = x^2 - 4x - 5$, factor
$0 = (x + 1)(x - 5)$

Gives $x = -1$ or $x = 5$

50. continued

So our two points on the parabola ($y = x^2 + x$) are:
For $x = -1$:
$y = (-1)^2 + 1 = 0$ → $(-1, 0)$
For $x = 5$:
$y = 5^2 + 5 = 30$ → $(5, 30)$

Now we go back to $y' = 2x + 1$ to get our actual slopes of the tangent lines:
For $x = -1$:
$y' = 2*(-1) + 1 = -1$
For $x = 5$:
$y' = 2*5 + 1 = 11$

So our first tangent line goes through (-1, 0) and (2, -3) with a slope of –1.
(Safety check notice: $(-3 - 0) / (2 - (-1)) = -1$, so it really has a slope of –1.
Using Point-Slope form of a line we get:
$(y - y1) = m*(x - x1)$
$(y - 0) = -1*(x - (-1))$
$y = -1*(x + 1)$
$y = -x - 1$

Our second tangent line goes through (5, 30) and (2, -3) with a slope of 11.
(Check: $(-3 - 30) / (2 - 5) = -33/-3 = 11$, so slope really is 11)
Using Point-Slope form of a line we get:
$(y - y1) = m*(x - x1)$
$(y - 30) = 11*(x - 5)$
$y - 30 = 11x - 55$
$y = 11x - 25$

The equations for the two lines which
go through the point (2, -3)
and are tangent to the parabola defined by $y = x^2 + x$ are:
$y = -x - 1$
and
$y = 11x - 25$

70. Where is the function h(x) = | x – 1 | + | x + 2 | differentiable? Give a formula for h' and sketch the graphs of h and h'.

Remember a function must be continuous and have no "sharp" turns to be differentiable. As this function involves absolute values it is likely to have some sharp turns. All that needs to be done is to find them.

Notice some odd things are likely to happen at x = 1 and x = -2.
Consider:

h(1) = 0 + 3 = 3
h(-2) = 3 + 0 = 3
h(0) = 1 + 2 = 3
h(1.5) = 0.5 + 2.5 = 3

In fact for $-2 \le x \le 1$, h(x) = 3.

What this means is that h(x) is NOT differentiable at x = 1 and x = -2, because there are "sharp" turns at those points.
So we conclude

$$\boxed{h(x) \text{ is differentiable on } x \in (-\infty, -2) \cup (-2, 1) \cup (1, \infty).}$$

As for the equation for h'(x). Break h(x) into a function on the above 3 intervals:

$$h(x) = \begin{cases} -(x-1)-(x+2) & x \le -2 \\ 3 & -2 < x < 1 \\ (x-1)+(x+2) & x \ge 1 \end{cases}$$

Simplifying a bit we get:

$$h(x) = \begin{cases} -2x-1 & x \le -2 \\ 3 & -2 < x < 1 \\ 2x+1 & x \ge 1 \end{cases}$$

And then we can easily arrive at the answer.

$$\boxed{h'(x) = \begin{cases} -2 & x < -2 \\ 0 & -2 < x < 1, \\ 2 & x > 1 \end{cases}}$$

leaving h' undefined at x = -2 and x = 1

The graphing is left for you to do.

72. Find the values of m and b that make the below function differentiable everywhere.

$$f(x) = \begin{cases} x^2 & x \le 2 \\ mx + b & x > 2 \end{cases}$$

Notice that because x^2 is a polynomial it is known to be differentiable everywhere (in this case that would be everywhere $x < 2$)
Likewise mx + b is a polynomial making f(x) differentiable everywhere $x > 2$.

So all we need to worry about is what happens at $x = 2$.
Specifically f(x) must be continuous at $x = 2$
AND there cannot be any sharp turns.

First thing to worry about:

To make f(x) continous at $x = 2$ all we need to do is find m and b such that:

$$\mathbf{x^2 = mx + b, \quad when\ x = 2}$$

Second thing to worry about:

To make sure that f(x) does NOT have any sharp turns we need m and b such that the derivative (f '(x)) is continuous at $x = 2$. So we find f '(x):

$$f'(x) = \begin{cases} 2x & x \le 2 \\ m & x > 2 \end{cases}$$

From this we see that we need

$$\mathbf{2x = m, \quad when\ x = 2}$$

Solve the two worries:

Now we solve the system of equations for m and b, putting 2 in for x:

$x^2 = mx + b$ → Equation #1: $4 = 2m + b$
$2x = m$ → Equation #2: $4 = m$

Use Equation #2 and put 4 in for m into Equation #1, and solve for b:

$4 = 2m + b$ → $4 = 2*4 + b$
→ $4 = 8 + b$
→ $-4 = b$

And there is our solution: m = 4 and b = -4

The values **m = 4** and **b = -4** will make $f(x) = \begin{cases} x^2 & x \le 2 \\ mx + b & x > 2 \end{cases}$ continuous everywhere.

Section 5.7

continuing from section 3.2

Solutions and Hints

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For this section you might want to refer to the below table:

Function Given	**Its Anti-Derivative**
x^n (and $n \neq -1$)	$\frac{x^{n+1}}{n+1}$
$\frac{1}{x}$	ln(| x |)
e^x	e^x
$\cos(x)$	$\sin(x)$
$\sin(x)$	$-\cos(x)$
$\sec^2(x)$	$\tan(x)$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1}(x)$
$\frac{1}{1+x^2}$	$\tan^{-1}(x)$

Section 5.7
3. Find the most general antiderivative of the function:

$f(x) = 6x^9 - 4x^7 + 3x^2 + 1$

So there are really four parts to this problem:
a. Find the antiderivative of $6x^9$.
b. Find the antiderivative of $-4x^7$.
c. Find the antiderivative of $3x^2$.
d. Find the antiderivative of $1 = x^0$.

All of these are solved by applying the "reverse" of the power rule:

Given $c*x^n$ its antiderivative is $c*\frac{x^{n+1}}{n+1}$

So $6x^9$ has the antiderivative: $6*\frac{x^{9+1}}{9+1} = \frac{6x^{10}}{10} = \frac{3x^{10}}{5}$

$-4x^7$ has the antiderivative: $-4*\frac{x^{7+1}}{7+1} = \frac{-4x^8}{8} = \frac{-x^8}{2}$

$3x^2$ has the antiderivative: $3*\frac{x^{2+1}}{2+1} = \frac{3x^3}{3} = x^3$.

$1 = x^0$ has the antiderivative: $1*\frac{x^{0+1}}{0+1} = x$.

And then you put it all together:

$f(x) = 6x^9 - 4x^7 + 3x^2 + 1$

means

$F(x) = \frac{3x^{10}}{5} + \frac{-x^8}{2} + x^3 + x$ and do NOT forget the + C

$$\boxed{F(x) = \frac{3x^{10}}{5} + \frac{-x^8}{2} + x^3 + x + C}$$

Section 5.7

9. Find the most general antiderivative of the function:

$$g(t) = \frac{t^3 + 2t^2}{\sqrt{t}} = \frac{t^3 + 2t^2}{t^{1/2}}$$

This one may look difficult, however begin by dividing everything out:

$$g(t) \quad = \frac{t^3 + 2t^2}{t^{1/2}} \quad = \frac{t^3}{t^{1/2}} + \frac{2t^2}{t^{1/2}}$$

$$= t^{3-(1/2)} + 2t^{2-(1/2)}$$

$$= t^{5/2} + 2t^{3/2}$$

And now apply the reverse power rule: Given c*x^n its antiderivative is c* $\frac{x^{n+1}}{n+1}$

The antiderivative of $t^{5/2}$ is: $\frac{t^{(5/2)+1}}{(5/2)+1} = \frac{t^{7/2}}{(7/2)} = t^{7/2} * \frac{2}{7} = \frac{2t^{7/2}}{7}$

The antiderivative of $2t^{3/2}$ is: $\frac{2t^{(3/2)+1}}{(3/2)+1} = \frac{2t^{5/2}}{(5/2)} = 2t^{5/2} * \frac{2}{5} = \frac{4t^{5/2}}{5}$

Now put it together, and remember the +C:

Given: $g(t) \quad = t^{5/2} + 2t^{3/2}$

Then $G(t) \quad = \frac{2t^{7/2}}{7} + \frac{4t^{5/2}}{5} + C$

$$\boxed{G(t) \quad = \frac{2t^{7/2}}{7} + \frac{4t^{5/2}}{5} + C}$$