Section 3.5
Solutions and Hints

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for the book:
Calculus, Early Vectors
by James Stewart.

This section covers the Chain Rule. If you think too hard about it, this will be confusing. One way I have found useful to think about it goes like this:

Say you have a function F(x) where there is obviously an "outside" function with "stuff" inside. For example:

$$F(x) = \sqrt{x^2 + 1}.$$

The "outside" function would be the square root and the "stuff" inside would be $x^2 + 1$. Let O(*stuff*), that's "oh of stuff," be the outside function.

Step 1:
Now we know how to find the derivative of O(*stuff*) = $\sqrt{stuff}$ = $(stuff)^{1/2}$.
That would just be: O'(*stuff*) = (½)* $(stuff)^{-1/2}$.

Step 2:
And we know how to take the derivative of *stuff* = $x^2 + 1$.
That would just be *stuff* **'** = 2x.

Answer:
And believe it or not that is all we need to do. Because F'(x) = O'(*stuff*) * *stuff* **'**.
Or more specifically

F'(x) = [(½)* $(stuff)^{-1/2}$] * 2x, notice here *stuff* = $x^2 + 1$ so we can put that back in to get:

F'(x) = [(½)* $(x^2 + 1)^{-1/2}$] * 2x

So in sum if you are given a function:
F(x) = O(*stuff*)
Then
F'(x) = O'(*stuff*) * (the derivative of *stuff*)

11. Find the derivative of the given function.

$$g(x) = \sqrt{x^2 - 7x}$$

Rewrite this as: $g(x) = (x^2 - 7x)^{1/2}$

So for this one our outer function is square root, O(*stuff*) = (*stuff*)$^{1/2}$.
And *stuff* = $x^2 - 7x$.
Our final answer will be O'(*stuff*) * (derivative of *stuff*)
Let's make a table:

O(*stuff*) = (*stuff*)$^{1/2}$	*stuff* = $x^2 - 7x$
O'(*stuff*) = ½*(*stuff*)$^{-1/2}$ O'(*stuff*) $= \left(\frac{1}{2}\right) * \left(\frac{1}{stuff^{1/2}}\right)$	*stuff* ' = derivative of *stuff* = 2x – 7

By the Chain Rule:
g'(x) = O'(*stuff*) * (derivative of *stuff*)

$= \left(\frac{1}{2}\right) * \left(\frac{1}{stuff^{1/2}}\right) * (2x - 7)$, and put $x^2 - 7x$ in for *stuff*

$= \left(\frac{1}{2}\right) * \left(\frac{1}{(x^2 - 7x)^{1/2}}\right) * (2x - 7)$, simplify and combine things

$$= \frac{2x - 7}{2\sqrt{x^2 - 7x}}$$

$$g'(x) = \frac{2x - 7}{2\sqrt{x^2 - 7x}}$$

26. Find the derivative of the given function.

$$y = \tan(x^2) + \tan^2(x)$$

<u>Finding the derivative of **$\tan(x^2)$**</u>:

Here the outer function O(stuff) = tan(stuff). And the stuff = x^2.

O(stuff)	= tan(stuff)	stuff	= x^2
O'(stuff)	= $\sec^2$(stuff)	stuff '	= 2x

Derivative of $\tan(x^2)$ = O'(stuff) * stuff '
= $\sec^2$(stuff) * 2x *Put x^2 in for stuff.*
= $\sec^2(x^2) * 2x$
= $\mathbf{2x * \sec^2(x^2)}$

<u>Finding the derivative of **$\tan^2(x)$**</u>:

Notice $\tan^2(x) = \tan(x) * \tan(x)$, so we could apply the Product Rule, but we may also apply the Chain Rule if you think like: $\tan^2(x) = [\ \tan(x)\ ]^2$

Let our outer function be "squared." So O(stuff) = $(\text{stuff})^2$. And stuff = tan(x).

O(stuff)	= $(\text{stuff})^2$	stuff	= tan(x)
O'(stuff)	= $2*(\text{stuff})^1$	stuff '	= $\sec^2(x)$

Derivative of $\tan^2(x)$ = O'(stuff) * stuff '
= [2*stuff] * $\sec^2(x)$, *put tan(x) in for stuff.*
= $\mathbf{2 * \tan(x) * \sec^2(x)}$
= $2 * \sin(x) / \cos^3(x)$

<u>Finding y'</u>:

$y = \tan(x^2) + \tan^2(x)$

y' = [derivative of $\tan(x^2)$] + [derivative of $\tan^2(x)$]

y' = [$2x * \sec^2(x^2)$] + [$2 * \tan(x) * \sec^2(x)$]

OR rewriting stuff: $y' = \dfrac{2x}{(\cos(x^2))^2} + \dfrac{2*\sin(x)}{(\cos(x))^3}$

OR rewriting stuff: $y' = \dfrac{2*\left(x*\cos^3(x) + \sin(x)*\cos^2(x^2)\right)}{\cos^2(x^2)*\cos^3(x)}$

$$y' = [\ 2x * \sec^2(x^2)\ ] + [\ 2 * \tan(x) * \sec^2(x)\ ]$$

34. Find the derivative of the given function.

This one requires the chain rule to be applied several times. So don't get confused.

y = sin^2(cos 4x)

The outer function O(stuff) = sin^2(stuff) and stuff = cos(4x)

O(stuff) = sin^2(stuff)   stuff = cos(4x)

Notice to find O' we need to apply the Chain Rule again:

O(stuff) = sin^2(stuff) = [sin(stuff)]^2

Let the outer function F(more_stuff) = (more_stuff)^2 and more_stuff = sin(stuff)

F(more_stuff) = (more_stuff)^2 more_stuff = sin(stuff)

F'(more_stuff) = 2*(more_stuff)   more_stuff ' = cos(stuff)

O'(stuff) = F'(more_stuff) * more_stuff '

= 2*more_stuff * cos(stuff),   *put sin(stuff) in for more_stuff*

= 2 * sin(stuff) * cos(stuff)

So now we have for y = sin^2(cos 4x):

The outer function O(stuff) = sin^2(stuff) and stuff = cos(4x)

O(stuff) = sin^2(stuff)   stuff = cos(4x)

O'(stuff) = 2 * sin(stuff) * cos(stuff)

But we need to apply the chain rule to find stuff '

stuff = cos(4x)

Let the outer function G(more_stuff) = cos(more_stuff) and more_stuff = 4x

G(more_stuff) = cos(more_stuff)   more_stuff = 4x

G'(more_stuff) = –sin(more_stuff)   more_stuff ' = 4

stuff ' = G'(more_stuff) * more_stuff '

= –sin(more_stuff) * 4,   *put in 4x for more_stuff.*

= –sin(4x)*4

= –4*sin(4x)

So now we have for y = sin^2(cos 4x):

The outer function O(stuff) = sin^2(stuff) and stuff = cos(4x)

O(stuff) = sin^2(stuff)   stuff = cos(4x)

O'(stuff) = 2 * sin(stuff) * cos(stuff)   stuff ' = –4*sin(4x)

And we arrive at:

y' = O(stuff) * stuff '

= [2 * sin(stuff) * cos(stuff)] * [–4*sin(4x)],   *put in cos(4x) for stuff*

= [2 * sin(cos(4x)) * cos(cos(4x))] * [–4 * sin(4x)]

= –8 * sin(cos(4x)) * cos(cos(4x)) * sin(4x)

y' = –8 * sin(cos(4x)) * cos(cos(4x)) * sin(4x)

60. Suppose that $w = u \circ v$ and $u(0) = 1$, $v(0) = 2$, $u'(0) = 3$, $u'(2) = 4$, $v'(0) = 5$ and $v'(2) = 6$. Find $w'(0)$.

Much of the information given is just for distraction.

$w = u \circ v$ is the same as saying: $w(x) = u(v(x))$

So the outer function is u(stuff) and the stuff = v(x).

O(stuff) = u(stuff)	stuff = v(x)
O'(stuff) = u'(stuff)	stuff ' = v'(x)

w'(x)	= O'(stuff) * stuff '	
	= u'(stuff) * v'(x)	and we can put v(x) in for stuff
	= u'(v(x)) * v'(x)	
Thus		
w'(0)	= u'(v(0)) * v'(0)	we just put zero in for x everywhere
		we are given v(0) = 2 and v'(0) = 5
	= u'(2) * 5	and we are given u'(2) = 4
	= 4 * 5	
	= 20	

w'(0) = 20