Section 3.6
Solutions and Hints

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for the book:

Calculus, Early Vectors
by James Stewart.

Read over and understand Example #1 and Example #2 of this section. They WILL help you understand implicit differentiation. Note that dy/dx and y' mean the same thing.

25. Find an equation of the tangent line to the curve at the given point.

$2*(x^2 + y^2)^2 = 25 * (x^2 - y^2)$ at the point (3, 1)

We need the value of y' when x = 3 and y = 1 for the slope of the tangent line. We could use the chain rule for the $(x^2 + y^2)^2$, but as it is only squared, multiplying it out is just as easy. So begin with:

$d/dx(2x^4 + 4x^2y^2 + 2y^4) = d/dx(25x^2 - 25y^2)$
$8x^3 + 8xy^2 + 8x^2yy' + 8y^3y' = 50x - 50yy'$ solve for y'
$8x^3 + 8xy^2 - 50x = -50yy' - 8x^2yy' - 8y^3y'$
$8x^3 + 8xy^2 - 50x = (-50y - 8x^2y - 8y^3) * y'$
$(8x^3 + 8xy^2 - 50x) / (-50y - 8x^2y - 8y^3) = y'$

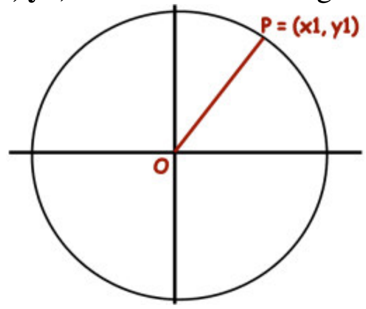
Put 3 in for x and 1 in for y:
$y' = (8x^3 + 8xy^2 - 50x) / (-50y - 8x^2y - 8y^3)$
$= (8*27 + 8*3 - 50*3) / (-50 - 8*9 - 8)$
$y' = -9/13$

Using the Point-Slope form of a line we have:
$(y - y1) = m*(x - x1)$
$(y - 1) = -9/13*(x - 3)$
$y - 1 = -9/13x + 27/13$
$y = -9/13x + 40/13$ rewriting we have $13y + 9x - 40 = 0$

$y = -9/13x + 40/13$ OR, rewriting $13y + 9x - 40 = 0$

33. Show, using implicit differentiation, that any tangent line at a point P to a circle with center O is perpendicular to the radius OP.

Without loss of generality (WLOG) we will **assume the center O of the circle to be at the origin (0, 0)**. Note most of the time O is used to mean that anyway, and it may have been meant here also.
Let P be the point (x1, y1) on the circle. This gives the picture:



So the slope of the radial line OP will always = y1/x1.
What we need to find is the slope of the tangent line at (x1, y1) and demonstrate it equals –x1/y1 (the negative reciprocal).

The length of the radius of the circle = r = $\sqrt{(x1-0)^2+(y1-0)^2} = \sqrt{(x1^2+y1^2}$
And the center (h, k) = (0,0).
Recall the equation of a circle is: $(x – h)^2 + (y – k)^2 = r^2$.
Thus our circle's equation is:

$(x – 0)^2 + (y – 0)^2 = (x1^2 + y1^2)$,
$x^2 + y^2 = x1^2 + y1^2$

Remember that x1 and y1 are constants. And use the above to find y' → which will tell us the slope of the tangent line at ANY point (x, y).

$d/dx(x^2 + y^2) = d/dx(x1^2 + y1^2)$
2x + 2y*y' = 0, *Solve for y'.*
2x = –2y*y'
$\frac{2x}{-2y}$ = y'
$\frac{x}{-y}$ = y'

Now all we need to do is put x1 in for x and y1 in for y: $\frac{x}{-y} = \frac{x1}{-y1}$

And so for any given point P = (x1, y1) we see that the slope of the radial line OP is y1/x1 and the slope of the tangent line at that point is –x1/y1. Thus the radial line and the tangent line to any point on any given circle are perpendicular.

35. Show the given curves are orthogonal.

$$2x^2 + y^2 = 3 \quad \text{and} \quad x = y^2$$

We will use implicit differentiation to find the equation of the slope for each function:

$d/dx(2x^2 + y^2) = d/dx(3)$ → $4x + 2y*y' = 0$
→ $y' = –4x / 2y$
→ $y' = –2x / y$

$d/dx(x) = d/dx(y^2)$ → $1 = 2y*y'$
→ $y' = 1 / 2y$

Notice that –2x / y is the negative reciprocal of 1 / 2y for any (x, y) pair.

Thus for any point (x1, y1) where the curves intersect, the tangent lines of the two curves will have perpendicular slopes. Thus, by definition, the curves are orthogonal.

ASIDE: In some instances of this type of question you will actually need to find the points where the curves intersect. The above is NOT one of those instances.