# Section 4.8
# Solutions and Hints

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## for the book:
**<u>Calculus, Early Vectors</u>**
**by James Stewart.**

**40. Find the $\lim_{x \to -\infty} x * e^x$.**

Recall that $\lim_{x \to \text{negative } \infty} e^x = 0$. And clearly $\lim_{x \to \text{negative } \infty} x = -\infty$.
So we have a case of $-\infty * 0$, which is NOT automatically equal 0 and notice that L'Hôpital's Rule does NOT apply, so we must change stuff up a bit:
The first thing we do is the ultimate in sneaky tricks:
We use the fact that $(-1) * (-1) = 1$

$$x * e^x = (-x) * (-e^x), \qquad \text{Now we use the fact } (-e^x) = \frac{1}{\left(\frac{1}{-e^x}\right)}$$

$$= (-x) * \frac{1}{\left(\frac{1}{-e^x}\right)}$$

So $\lim_{x \to -\infty} x * e^x = \lim_{x \to -\infty} \frac{-x}{\left(\frac{1}{-e^x}\right)}$

Notice this gives us $\infty / \infty$. Had we not originally multiplied twice by $-1$, we would now be looking $-\infty / \infty$, and we would NOT be able to apply L'Hôpital's Rule. Fortunately we did so we can:

$$\lim_{x \to -\infty} x * e^x = \lim_{x \to -\infty} \frac{-x}{\left(\frac{1}{-e^x}\right)}, \quad \textit{Take derivative of } -x \textit{ and } (1 / -e^x).$$

$$= \lim_{x \to -\infty} \frac{-1}{\left(\frac{1}{e^x}\right)}$$

$$= \lim_{x \to -\infty} -e^x, \qquad \textit{by "definition" } \lim_{x \to -\infty} e^x = 0.$$

$$= 0$$

$\lim_{x \to -\infty} x * e^x = 0$

**61. Find the $\lim_{x \to \infty} x^{1/x}$.**

Notice as written we have $\infty^0$, which is indeterminate, which means we will likely need to use L'Hôpital's Rule somewhere.

Because this problem has x in the base and exponent it is likely we will use the natural log in some way…

Let $y = x^{1/x}$.

Thus $\lim_{x \to \infty} x^{1/x} = \lim_{x \to \infty} y$
$= \lim_{x \to \infty} e^{\ln y}$.

So if we found the $\lim_{x \to \infty} \ln y$ then we would know $\lim_{x \to \infty} x^{1/x}$.

$$\begin{aligned}
\lim_{x \to \infty} \ln y &= \lim_{x \to \infty} \ln( x^{1/x} ) \\
&= \lim_{x \to \infty} (1/x) * \ln(x) \\
&= \lim_{x \to \infty} \frac{\ln(x)}{x}, \qquad \textit{which is } \infty/\infty\textit{, so apply L.H.} \\
&= \lim_{x \to \infty} \frac{\left(\frac{1}{x}\right)}{1} \\
&= \lim_{x \to \infty} 1/x \\
&= 0
\end{aligned}$$

We are NOT done yet, we must remember where we started:

$$\begin{aligned}
\text{Thus} \quad \lim_{x \to \infty} x^{1/x} &= \lim_{x \to \infty} y \\
&= \lim_{x \to \infty} e^{\ln y}, \qquad \textit{and now put 0 in from above} \\
&= e^0 \\
&= 1
\end{aligned}$$

$\lim_{x \to \infty} x^{1/x} = 1$