Section 5.3
Solutions and Hints

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for the book:

Calculus, Early Vectors
by James Stewart.

16a. Find the intervals on which f is increasing or decreasing.

$f(x) = x\sqrt{x+1} = x * (x+1)^{1/2}$, Notice this is undefined for $x < -1$

f(x) is increasing when f '(x) > 0 and decreasing when f '(x) < 0.

So find f(x) and then solve the two inequalities: f '(x) > 0 and f '(x) < 0.

Using the Product Rule: f(x) = g(x) * h(x)

g(x) = x h(x) = $(x + 1)^{1/2}$

Apply the Chain rule to find h'(x):

O(stuff) = $(\text{stuff})^{1/2}$ stuff = x + 1

O'(stuff) = ½*$(\text{stuff})^{-1/2}$ stuff ' = 1

g'(x) = 1 h'(x) = ½*$(x+1)^{-1/2}$

f '(x) = g(x)*h'(x) + g'(x)*h(x)

= x * [½ *$(x + 1)^{-1/2}$ + 1 * $(x + 1)^{1/2}$

$= \dfrac{x}{2\sqrt{x+1}} + \sqrt{x+1}$, *Notice this is undefined for $x \le -1$*

So **f '(x) does not exist at x = -1**. We now find the points where f '(x) = 0

$\dfrac{x}{2\sqrt{x+1}} + \sqrt{x+1} = 0$ → $\dfrac{x}{2\sqrt{x+1}} = -\sqrt{x+1}$

→ x = –2*(x + 1)

→ x = –2x – 2

→ 3x = –2

→ x = –2 / 3

f '(x) = 0 at x = –2/3.

So we have critical points at x = –1 and x = –2/3

and everything is undefined for x < -1.

Therefore we must check f '(x) on the intervals (–1, –2/3) and (–2/3, ∞)

f '(–7/8) ≅ –0.8838834765 and f '(3) = 4.75

Thus f is decreasing on (–1, –2/3) and increasing on (–2/3, ∞).

16b. Find the local maximum and minimum values of f.

From part (a) we know the critical points occur at x = –1 and x = –2/3, so if there are any extrema they must occur there.

f(-1) = $x\sqrt{x+1}$ = -1*0 = 0

f(-2/3) ≅ –0.3849001795

So it is likely f(-1) is a local maximum and f(-2/3) is a local minimum.

Notice from part (a) we can also conclude that f(-2/3) must be a global minimum.

16c. Find the intervals of concavity and the inflection points.

From part (a) we have:

$$f'(x) = \frac{x}{2\sqrt{x+1}} + \sqrt{x+1}$$

Finding the derivative of $\frac{x}{2\sqrt{x+1}}$:

Let g(x) = x and h(x) = $\frac{1}{2} * (x+1)^{-1/2}$

Then g'(x) = 1 and h'(x) = $-\frac{1}{4} * (x+1)^{-3/2}$

$$\frac{d}{dx}\left(\frac{x}{2\sqrt{x+1}}\right) = g(x)*h'(x) + g'(x)*h(x)$$

$$= x*(-\tfrac{1}{4}(x+1)^{-3/2}) + 1*(\tfrac{1}{2}*(x+1)^{-1/2})$$

$$= \frac{-x}{4(x+1)^{3/2}} + \frac{1}{2(x+1)^{1/2}}$$, *common denominator is* $4(x+1)^{3/2}$.

$$= \frac{-x + 2*(x+1)^{2/2}}{4*(x+1)^{3/2}}$$

$$= \frac{x+2}{4*(x+1)^{3/2}}$$

Which gives:

$$f''(x) = \frac{x+2}{4(x+1)^{3/2}} + \frac{1}{2\sqrt{x+1}}$$

Notice this cannot be zero anywhere as we must be taking the square roots of positive numbers (by x > -1) and there is no subtraction anywhere. Thus we further deduce that f ''(x) is > 0 everywhere, for x > -1.

So f(x) is concave up on (-1, ∞)

16d. Using (a), (b) and (c) sketch the graph of f.

This is left for you to do.

45. Two runners start a race at the same time and finish in a tie. Prove that at some time during the race they have the same velocity.

Let g(t) = the position of the first runner at time t.
thus g'(t) is his velocity.
Let h(t) = the position of the second runner at time t.
thus h'(t) is her velocity.
Let t_0 = their start time.
Let t_f = their finish time.
Notice $g(t_0) = h(t_0)$ and $g(t_f) = h(t_f)$

Consider f(t) = g(t) – h(t).

We want to show that at some $t \in (t_0, t_f)$ that g'(t) = h'(t).
Notice f '(t) = g'(t) – h'(t), as we have defined f(t).
So if we can show that f'(t) is zero for some $t \in (t_0, t_f)$ we will have shown that there is a time during the race where the runners had the same velocity.

Because $g(t_0) = h(t_0)$ we know $f(t_0) = 0$.
Likewise because $g(t_f) = h(t_f)$ we know $f(t_f) = 0$.

Assume f '(t) never equals zero. Then there are two cases:
Case 1:
f(t) is strictly increasing (i.e. f '(t) > 0 for all t).
Then for all $t > t_0$ $f(t) > f(t_0)$.
However we know that $f(t_0) = f(t_f)$.
Thus we have a contradiction so this case cannot happen.
Case 2:
f(t) is strictly decreasing (i.e. f '(t) < 0 for all t).
Then for all $t > t_0$ $f(t) < f(t_0)$.
However we know that $f(t_0) = f(t_f)$.
Thus we have a contradiction so this case cannot happen.

Since neither case 1 nor case 2 can be true our assumption that f '(t) never equals zero must be false. Thus there must exist some c for which f '(c) = 0.
Thus there exists a time = c when the runners have the same velocity.