Section 5.5
Solutions and Hints

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for the book:
Calculus, Early Vectors
by James Stewart.

6. A farmer wants to fence an area of 1.5 million square feet in a rectangular field and then divide it in half with a fence parallel to one of the sides of the rectangle. How can he do this so as to minimize the cost of the fence? (Assume cost is directly proportional to the length of fence)

Let x = length of two sides of the fence
Let y = length of two sides of the fence and the 'dividing' fence.

Area = A = x * y = 1.5 million square feet

Total length of fencing used L = 2x + 3y

We want to minimize L (and thus minimize cost) subject to xy = 1.5 million

L = 2x + 3y *and we know xy = 1.5 → y = 1.5/x, so sub that in.*
L = 2x + 3* (1.5/x) *simplify.*
$L = 2x + 4.5 * x^{-1}$

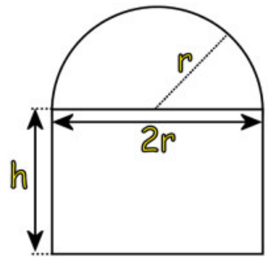
$L' = 2 - 4.5 * x^{-2}$ *To find min/max set the first derivative = 0 and solve for x.*

$L' = 2 - \frac{4.5}{x^2} = 0$ → $2 = \frac{4.5}{x^2}$
→ $2x^2 = 4.5$
→ $x^2 = 2.25$
→ x = 1.5 *x is a distance so we keep only the positive.*

And since y = 1.5/x → y = 1.5/1.5 → y = 1

So the dimensions 1.5 million feet by 1 million feet will minimize the cost of the fence, subject to the given description.

28. A Norman window has the shape of a rectangle surmounted by a semicircle – thus the diameter of the semicircle is equal to the width of the rectangle. If the perimeter of the window is 30 feet, find the dimensions of the window which allow the greatest amount of light to be admitted.



We are given the perimeter is 30 feet.
Recall the circumference of a circle is π*(diameter), and in this case we are dealing with just half of a circle so we have the curved part of the window = $\frac{1}{2}*\pi*(2r) = r*\pi$.
The outside part of the rectangle is h + 2r + h = 2r + 2h. Thus **$30 = r*\pi + 2r + 2h$**.
The area of a circle is $\pi*r^2$, but in this case we only need half of a circle, so the area of the curved top part of the window is $\frac{1}{2} * \pi * r^2$. And the area of the rectangle is 2r * h.
So the total area = **$A = \frac{1}{2}*\pi*r^2 + 2r*h$**.
In sum we know:

Equation 1: $30 = r*\pi + 2r + 2h$ and Equation 2: $\text{Area} = \frac{1}{2}*\pi*r^2 + 2rh$

We will use equation 1 to solve for h in terms of r:

$30 = r*\pi + 2r + 2h \rightarrow 30 - r*\pi - 2r = 2h$

$\rightarrow 15 - \frac{1}{2}*r*\pi - r = h$ *(call this equation 3)*

So we substitute $(15 - \frac{1}{2}*r*\pi - r)$ in for h into equation 2:

$A = \frac{1}{2}*\pi*r^2 + 2rh \rightarrow A = \frac{1}{2}*\pi*r^2 + 2r*(15 - \frac{1}{2}*r*\pi - r)$

$\rightarrow A = \frac{1}{2}*\pi*r^2 + 30r - r^2*\pi - 2r^2$

$\rightarrow A = 30r - \frac{1}{2}*r^2\pi - 2r^2$

As we need to maximize area we take the derivative of A, set it equal to 0 and solve for r:

$A = 30r - \frac{1}{2}*r^2\pi - 2r^2 \rightarrow A' = 30 - r*\pi - 4r$

$30 - r*\pi - 4r = 0 \rightarrow 30 = r*\pi + 4r$

$\rightarrow 30 = r * (\pi + 4)$

$\rightarrow \frac{30}{\pi + 4} = r$

We now use this value for r and put it back into equation 3 to find h:

$15 - \frac{1}{2}*r*\pi - r = h \rightarrow 15 - \frac{15}{\pi + 4}*\pi - \frac{30}{\pi + 4} = h$

$\rightarrow h = \frac{30}{\pi + 4}$

So the width of the window = $2r = \frac{60}{\pi + 4}$ feet and height = $\frac{30}{\pi + 4}$ feet.