Section 5.7
Solutions and Hints
by Brent M. Dingle
for the book:
Calculus, Early Vectors
by James Stewart.

For this section you might want to refer to the below table:

Function Given	Its Anti-Derivative
$k * x^n$ (and $n \neq -1$) *where k is a constant*	$\frac{k * x^{n+1}}{n+1}$ *where k is a constant*
$\frac{1}{x}$	ln(| x |)
k * e^x *where k is a constant*	k * e^x *where k is a constant*
k * cos(x) *where k is a constant*	k * sin(x) *where k is a constant*
k * sin(x) *where k is a constant*	-k * cos(x) *where k is a constant*
k * $\sec^2(x)$ *where k is a constant*	k * tan(x) *where k is a constant*
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1}(x)$
$\frac{1}{1+x^2}$	$\tan^{-1}(x)$

3. Find the most general antiderivative of the function:

$f(x) = 6x^9 - 4x^7 + 3x^2 + 1$

So there are really four parts to this problem:

a. Find the antiderivative of $6x^9$.
b. Find the antiderivative of $-4x^7$.
c. Find the antiderivative of $3x^2$.
d. Find the antiderivative of $1 = x^0$.

All of these are solved by applying the "reverse" of the power rule:

Given $c*x^n$ its antiderivative is $c*\frac{x^{n+1}}{n+1}$

So $6x^9$ has the antiderivative: $6*\frac{x^{9+1}}{9+1} = \frac{6x^{10}}{10} = \frac{3x^{10}}{5}$

$-4x^7$ has the antiderivative: $-4*\frac{x^{7+1}}{7+1} = \frac{-4x^8}{8} = \frac{-x^8}{2}$

$3x^2$ has the antiderivative: $3*\frac{x^{2+1}}{2+1} = \frac{3x^3}{3} = x^3$.

$1 = x^0$ has the antiderivative: $1*\frac{x^{0+1}}{0+1} = x$.

And then you put it all together:

$f(x) = 6x^9 - 4x^7 + 3x^2 + 1$

means

$F(x) = \frac{3x^{10}}{5} + \frac{-x^8}{2} + x^3 + x$ and do NOT forget the + C

$$F(x) = \frac{3x^{10}}{5} + \frac{-x^8}{2} + x^3 + x + C$$

9. Find the most general antiderivative of the function:

$$g(t) = \frac{t^3 + 2t^2}{\sqrt{t}} = \frac{t^3 + 2t^2}{t^{1/2}}$$

This one may look difficult, however begin by dividing everything out:

$$g(t) = \frac{t^3 + 2t^2}{t^{1/2}} = \frac{t^3}{t^{1/2}} + \frac{2t^2}{t^{1/2}}$$

$$= t^{3-(1/2)} + 2t^{2-(1/2)}$$

$$= t^{5/2} + 2t^{3/2}$$

And now apply the reverse power rule: Given c*x^n its antiderivative is c*$\frac{x^{n+1}}{n+1}$

The antiderivative of $t^{5/2}$ is: $\frac{t^{(5/2)+1}}{(5/2)+1} = \frac{t^{7/2}}{(7/2)} = t^{7/2} * \frac{2}{7} = \frac{2t^{7/2}}{7}$

The antiderivative of $2t^{3/2}$ is: $\frac{2t^{(3/2)+1}}{(3/2)+1} = \frac{2t^{5/2}}{(5/2)} = 2t^{5/2} * \frac{2}{5} = \frac{4t^{5/2}}{5}$

Now put it together, and remember the +C:

Given: $g(t) = t^{5/2} + 2t^{3/2}$

Then $G(t) = \frac{2t^{7/2}}{7} + \frac{4t^{5/2}}{5} + C$

$$G(t) = \frac{2t^{7/2}}{7} + \frac{4t^{5/2}}{5} + C$$

11. h(x) = sin(x) – 2*cos(x)

So h(x) = f(x) + g(x), where f(x) = sin(x) and g(x) = –2 * cos(x)

H(x) = F(x) + G(x) + C

f(x) = sin(x) → F(x) = –cos(x), from looking at the table above
g(x) = –2*cos(x) → G(x) = –2*sin(x), from looking at the table above

Thus,
H(x) = F(x) + G(x) + C
= –cos(x) + (–2)*sin(x) + C
= –cos(x) – 2sin(x) + C

Do NOT forget the +C !!!

The antiderivative of h(x) = H(x) = –cos(x) – 2sin(x) + C

12. f(t) = sin(t) – 2*t$^{1/2}$.

Notice we rewrote $2\sqrt{t}$ as $t^{1/2}$. And as with problem 11 we see we can break the given function into the addition of two separate functions:

f(t) = g(t) + h(t), where g(t) = sin(t) and h(t) = –2*t$^{1/2}$.
F(t) = G(t) + H(t) + C

g(t) = sin(t) → G(t) = -cos(t)
h(t) = –2*t$^{1/2}$ → H(t) = (–2 * t$^{1/2+1}$) / (½ + 1) = (–2 * t$^{3/2}$) / (3/2) = (–4/3)*t$^{3/2}$.

Thus,
F(t) = G(t) + H(t) + C
F(t) = –cos(t) + (–4/3)*t$^{3/2}$ + C
= –cost(t) – (4/3)*t$^{3/2}$ + C

The antiderivative of f(t) = F(t) = –cost(t) – (4/3) * t$^{3/2}$ + C.

19. Find f(x) given f ''(x) = 1

The only tricky part of doing these is remembering to include the unknown constants (i.e. the +C, +D, etc) as you go.

Notice that f ''(x) = 1 is the same thing as f ''(x) = $1*x^0$.

f ''(x) = 1 → f '(x) = x + C, *where C is some constant.*

f '(x) = x + C → f(x) = $(½)x^2$ + Cx + D, *where D is some constant.*

f(x) = $(½)x^2 + Cx + D$

23. Find f(x) given f '(x) = 4x + 3 and f(0) = –9

Here since they give us a value of f(0) we will be able to solve for C.

$$f'(x) = 4x + 3 \rightarrow f(x) = \frac{4 * x^{1+1}}{1+1} + \frac{3 * x^{0+1}}{0+1} + C$$

→ f(x) = $2x^2 + 3x + C$

As we are given f(0) = –9 we can put 0 in for x and get:

f(0) = $2*0^2 + 3*0 + C$, *and put –9 in for f(0)*
–9 = 0 + 0 + C
–9 = C

Now put that back into our general solution for f(x):

f(x) = $2x^2 + 3x + C$ → f(x) = $2x^2 + 3x – 9$

f(x) = $2x^2 + 3x – 9$

34. Find f(x) given f ''(x) = x + $x^{1/2}$, f(1) = 1, f '(1) = 2

As we are given f(1) =1 and f '(1) = 2 we will be able to solve for C and D.

Notice we have rewritten $\sqrt{x}$ as $x^{1/2}$.

f ''(x) = x + $x^{1/2}$ → f '(x) = $\frac{1*x^{1+1}}{1+1} + \frac{1*x^{(1/2)+1}}{(1/2)+1} + C$

→ f '(x) = $x^2 + \frac{x^{3/2}}{(3/2)} + C$

→ f '(x) = $x^2 + \frac{2x^{3/2}}{3} + C$

Given f '(1) = 2 we can solve for C:

f '(1) = $1^2 + \frac{2*1^{3/2}}{3} + C$, *put 2 in for f '(1).*

2 = (5/3) + C, *solve for C.*

(1/3) = C

Thus

f '(x) = $x^2 + \frac{2x^{3/2}}{3} + C$ → f '(x) = $x^2 + \frac{2x^{3/2}}{3} + \frac{1}{3}$

Now we find f(x)

f '(x) = $x^2 + \frac{2x^{3/2}}{3} + \frac{1}{3}$ → f(x) = $\frac{1*x^{2+1}}{2+1} + \frac{(2/3)x^{3/2+1}}{(3/2)+1} + \frac{(1/3)*x^{0+1}}{0+1} + D$

→ f(x) = $\frac{x^3}{3} + \frac{4x^{5/2}}{15} + \frac{1}{3}x + D$

Given that f(1) = 1 we can solve for D:

f(1) = $\frac{1^3}{3} + \frac{4*1^{5/2}}{15} + \frac{1}{3}*1 + D$, *put 1 in for f(1).*

1 = (1/3) + (4/15) + (1/3) + D

(1/15) = D

Thus

f(x) = $\frac{x^3}{3} + \frac{4x^{5/2}}{15} + \frac{1}{3}x + D$ → f(x) = $\frac{x^3}{3} + \frac{4x^{5/2}}{15} + \frac{1}{3}x + \frac{1}{15}$

f(x) = $\frac{x^3}{3} + \frac{4x^{5/2}}{15} + \frac{1}{3}x + \frac{1}{15}$

58. A particle is moving according to the given equations. Find the position of the particle.

Given: $a(t) = \cos(t) + \sin(t)$ and $s(0) = 0$ and $v(0) = 5$

For this $a(t)$ is acceleration, $v(t)$ is velocity, $s(t)$ is position.
AND $s''(t) = a(t)$ AND $s'(t) = v(t)$

Effectively we are given:

$$\mathbf{s''(t) = \cos(t) + \sin(t), \quad s(0) = 0, \quad s'(0) = 5}$$

And everything works as problems 23 to 40.

$s''(t) = \cos(t) + \sin(t) \rightarrow s'(t) = \sin(t) - \cos(t) + C$

Given $s'(0) = 5$ we can solve for C:

$s'(0) = \sin(0) - \cos(0) + C$
$5 = 0 - 1 + C$
$6 = C$

Thus

$s'(t) = \sin(t) - \cos(t) + C \rightarrow s'(t) = \sin(t) - \cos(t) - 6$

From that we can find $s(t)$:
$s'(t) = \sin(t) - \cos(t) - 6 \rightarrow s(t) = -\cos(t) - \sin(t) - 6t + D$

Given $s(0) = 0$ we can solve for D:

$s(0) = -\cos(0) - \sin(0) - 6*0 + D$
$0 = -1 - 0 - 0 + D$
$1 = D$

And we have
$s(t) = -\cos(t) - \sin(t) - 6t + D \rightarrow s(t) = -\cos(t) - \sin(t) - 6t + 1$

$s(t) = -\cos(t) - \sin(t) - 6t + 1$

73. A stone was dropped off a cliff and hit the ground with a speed of 120 feet per second. What is the height of the cliff ?

Here you need to know quite a few things:
Let a(t) be acceleration, v(t) be velocity and s(t) be position.
acceleration = a(t) = –32 ft/s^2 (this is just how physics on earth is)
(the negative sign indicates downward)
s''(t) = v'(t) = a'(t)

Let t_{final} be the time in seconds at which the stone hits the ground.
Let s(0) = 0 (sea level if you will)
Let v(0) = s'(0) = 0 (as it was simply dropped, it has no initial velocity)
Thus | s(t_{final}) | = the height of the cliff.

We begin with a(t) = s''(t) and use anti-differentiation to find v(t) and s(t).

a(t) = s''(t) = –32 → v(t) = s'(t) = –32*t + C

We know v(0) = 0 so

v(0) = –32*0 + C → 0 = 0 + C → C = 0

Thus

v(t) = –32*t

v(t) = s'(t) = –32*t → s(t) = –16*t^2 + D

We know s(0) = 0 so

s(0) = –16*0^2 + D → 0 = 0 + D → D = 0

Thus

s(t) = –16*t^2

In sum we have:

a(t) = –32
v(t) = –32*t
s(t) = –16*t^2

We need to find t_{final} when the stone hit the ground.
We know at this time its speed was 120 ft/s in the downward direction thus:

v(t_{final}) = –120 = –32*t_{final} → $\frac{-120}{-32} = t_{final}$
→ t_{final} = 3.75 seconds

And we can now put 3.75 into the position equation to see how far the stone fell:

s(3.75) = –16*$(3.75)^2$ = –225

So the stone fell downward 225 feet, thus the cliff must be 225 feet high.

The cliff is 225 feet high.

80. A projectile is fired with an initial speed of 500 m/s and an angle of elevation 30° from a position of 200 meters above the ground.

80a. Find the range (horizontal distance) of the projectile.

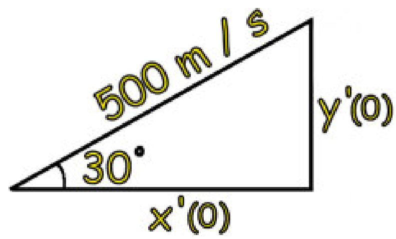
Let ground level be at height = 0.
Let x(t) be the horizontal position of the projectile at time t.
Let y(t) be the vertical position of the projectile at time t.
Thus x'(t) is the horizontal velocity and y'(t) is the vertical velocity.
Likewise x''(t) is the horizontal acceleration and y''(t) is the vertical acceleration.
Notice y(0) = 200 as the projectile was 200 meters above the ground when fired.
(y(0) = 0 for problem 79)
Let x(0) = 0 as this makes the horizontal distance easier to calculate.

<u>The Plan</u>:

To find the range we must find the time, t_{final}, such that $y(t_{final}) = 0$.
We will the put t_{final} into x(t) to find how far the projectile went horizontally.

<u>The Work</u>:

We are given the projectile was fired at 500 m/s at 30° elevation:



Thus the x'(0) = cos(30°) * 500 → $x'(0) = 250\sqrt{3}$ m/s
and y'(0) = sin(30°)*500 → y'(0) = 250 m/s

Let $y''(t) = -9.8 \text{ m/s}^2$, *this is just physics on the earth (the negative means down).*

Now we need to use anti-differentiation to find some stuff:

$y''(t) = -9.8 \rightarrow y'(t) = -9.8t + C$

Given y'(0) = 250 we get:

$y'(0) = -9.8*0 + C \rightarrow 250 = 0 + C \rightarrow C = 250$

So

$\mathbf{y'(t) = -9.8t + 250} \rightarrow y(t) = -4.9t^2 + 250t + D$

Given y(0) = 200 we get:

$y(0) = -4.9*0^2 + 250 * 0 + D \rightarrow 200 = 0 + 0 + D \rightarrow D = 200$

Thus

$\mathbf{y(t) = -4.9t^2 + 250t + 200}$

Recall we are trying to find the time, t_{final}, such that $y(t_{final}) = 0$.

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80a. continued

So set y(t) = 0 and solve for t:

$-4.9t^2 + 250t + 200 = 0$, *apply the quadratic equation to solve for t.*

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-250 \pm \sqrt{250^2 - 4*(-4.9)*200}}{2*(-4.9)}$$

$$= \frac{-250 \pm \sqrt{66420}}{-9.8}$$

$$\cong -0.787834606 \quad \text{or} \quad 51.80824277$$

As we cannot go backwards in time we discard the negative solution.
Thus the projectile will hit the ground roughly 51.8 seconds after it is fired.

We now must find x(t) and put 51.8 into it to find how far horizontally the projectile goes (its range).

Notice we will neglect wind resistance and friction and such so that x''(t) = 0.
x''(t) = 0 → x'(t) = 0 + C
From our picture above we discovered $x'(0) = 250\sqrt{3}$

$x'(0) = 0 + C \rightarrow 250\sqrt{3} = 0 + C \rightarrow C = 250\sqrt{3}$

So

$x'(t) = 250\sqrt{3} \rightarrow x(t) = 250\sqrt{3} * t + D$

We set x(0) = 0, thus:

$x(0) = 250\sqrt{3} *0 + D \rightarrow 0 = 0 + D \rightarrow D = 0$

So

$x(t) = 250\sqrt{3} * t$

And we know from our above work the projectile hits the ground roughly 51.8 seconds after it is fired so we put 51.8 into x(t):

$x(51.8) = 250\sqrt{3} * 51.8 \cong 22430.05796$

We conclude the projectile has a range of about 22,430 feet.

80b. Find the maximum height reached by the projectile.

The projectile will achieve its maximum height when its velocity switches from positive (moving upward) to negative (moving downward). Thus the maximum will occur at the time t_{max} such that $v_{vertical}(t_{max}) = 0$. Or rather we must solve $y'(t) = 0$ for t:

From part (a) we know **$y'(t) = -9.8t + 250$**. Set this = 0 and solve for t:

$$-9.8t + 250 = 0 \quad \rightarrow 250 = 9.8t$$
$$\rightarrow t \cong 25.51020408$$

So at roughly $t_{max} = 25.51$ seconds the projectile is at its highest point.

Thus if we put 25.5102 into y(t) we will find how high it went. From part (a) we know **$y(t) = -4.9t^2 + 250t + 200$**.

$$y(25.5102) = -4.9*(25.5102)^2 + 250 * (25.5102) + 200$$
$$\cong 3388.77551$$

The maximum height the projectile reaches is about 3389 feet.

80c. Find the speed of the projectile when it impacts the ground.

Speed is the magnitude of velocity.
Notice the projectile has both horizontal and vertical velocity.

From part (a) we know the projectile hits the ground at time $t \cong 51.8$ seconds. We put this into x'(t) and y'(t).

$$x'(51.8) = 250\sqrt{3} \quad \cong 433.0127019$$
$$y'(51.8) = -9.8*51.8 + 250 \quad \cong -257.64$$

We now apply the Pythagorean theorem to find the magnitude of the velocity:

$$\text{speed} = \sqrt{x'(51.8)^2 + y'(51.8)^2} = \sqrt{\left(250\sqrt{3}\right)^2 + (-257.64)^2}$$
$$\cong 503.8634434$$

The speed of the projectile upon impact is about 503.9 m/s.